# A Min-Plus / SDDP Algorithm for Deterministic Multistage Convex Programming

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# Dynamic Programming and Bellman operators

Given an integer $T > 0$, consider the Dynamic Programming equations

$$\begin{cases} V_T = \psi \\ \forall t \in [\![0, T-1]\!],\ V_t = \mathcal{B}_t(V_{t+1}) \end{cases}$$

where

- $\Psi$ is a function called the final cost function
- $\mathcal{B}_t$ is an operator called the Bellman operator
- $V_t$ is called the value function at time $t \in [\![0, T]\!]$
- We want to compute $V_0(x_0)$ at some given state $x_0$

# Deterministic Multistage (Convex) Programming (MSP)

Consider the Deterministic Multistage optimization problem

$$\min_{(x,u)} \sum_{t=0}^{T-1} c_t\left(x_t, u_t\right) + \psi\left(x_T\right)$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!],$$

$$x_{t+1} = f_t\left(x_t, u_t\right), \quad x_0 \text{ given.}$$

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Deterministic Multistage optimization problems can be solved by Dynamic Programming by setting $\mathcal{B}_t\left(\phi\right)\left(x\right) = \min_u c_t\left(x, u\right) + \phi\left(f_t\left(x, u\right)\right)$ (Bellman operator) with final cost $V_T := \psi$.

# What we will do

Build an algorithm that builds approximations of the value functions $V_t$ based on properties of the Bellman operators $\mathcal{B}_t$, e.g. monotonicity

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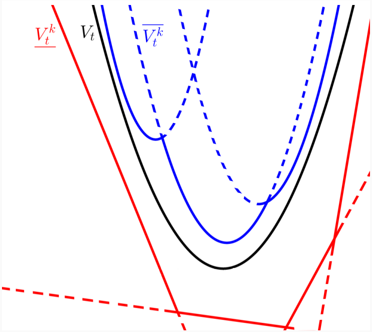
It must generalize the Stochastic Dual Dynamic Programming (SDDP) algorithm (developed by Pereira and Pinto 1991, Shapiro 2011, ...)
and the Min-plus algorithm for deterministic control problems (developed by McEneaney 2007, Qu 2014)

# Overview of our algorithm




Lower approximations $\underline{V_t}^k$ as a supremum of basic functions (affine functions for SDDP) below $V_t$

Upper approximations $\overline{V_t}^k$ as an infimum of some **other** basic functions (quadratic functions for Min-Plus) above $V_t$

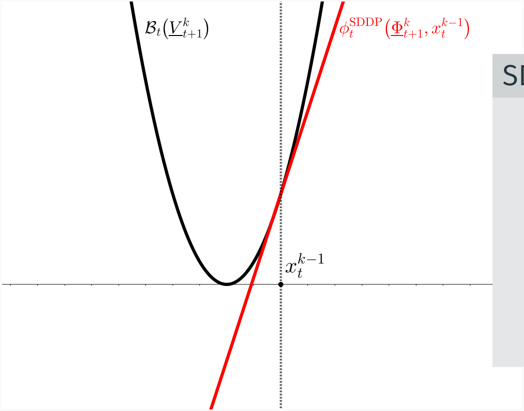
# Outline

1. Tropical Dynamic Programming (TDP): an algorithm encompassing both SDDP and a Min-Plus algorithm

2. Convergence result of TDP

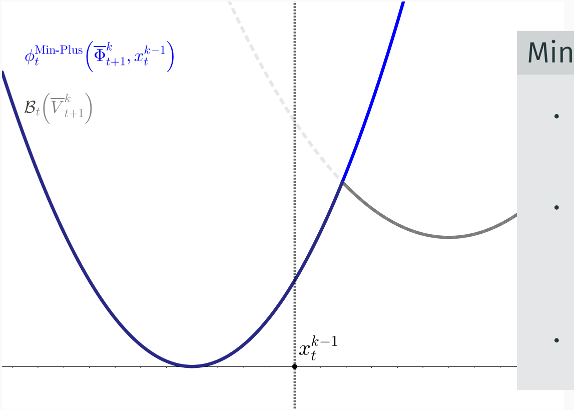
3. Numerical example: deterministic linear-quadratic optimal control with one constrained control

# Section content




### SDDP Exemple

- Affine functions
- Lower approximations opt $=$ sup
- $\underline{V}_{t+1}^k := \sup_{\phi \in \underline{\Phi}_{t+1}^k} \phi$




### Min-Plus Exemple

- Quadratic functions
- Upper approximations opt = inf
- $\overline{V}_{t+1}^k := \inf_{\phi \in \overline{\Phi}_{t+1}^k} \phi$

# Tight and Valid selection functions

## Tightness Assumption

$$\underbrace{\overbrace{\phi_t^{\text{SDDP}}}^{\text{Selection function}}\left(\overbrace{\underline{\Phi}_{t+1}^{k}}^{\text{Set of basic functions}}, x_t^{k-1}\right)}_{\text{Basic function}}(\overbrace{x_t^{k-1}}^{\text{Trial point}}) = \mathcal{B}_t\left(\underline{V}_{t+1}^{k}\right)\left(x_t^{k-1}\right)$$

It is a local property.

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## Validity Assumption

$$\phi_t^{\text{SDDP}}\left(\underline{\Phi}_{t+1}^k, x_t^{k-1}\right) \leq \mathcal{B}_t\left(\underline{V}_{t+1}^k\right) \quad \text{(SDDP)} \quad \textbf{opt} = \textbf{sup}$$

$$\phi_t^{\text{Min-Plus}}\left(\overline{\Phi}_{t+1}^k, x_t^{k-1}\right) \geq \mathcal{B}_t\left(\overline{V}_{t+1}^k\right) \quad \text{(Min-Plus)} \quad \textbf{opt} = \textbf{inf}$$

It is a global property.

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$$\Phi_t^{k+1} = \Phi_t^k \cup \{\phi\} .$$

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$$\Phi_t^{k+1} = \Phi_t^k \cup \{\phi\} \,.$$
5. **Update:** knowing the updated set of approximations $\left(\Phi_t^{k+1}\right)_t$ compute a new probability law $\mu^{k+1}$.

2. Convergence result of TDP

2.1 Almost sure uniform convergence to a limit $V_t^*$

2.2 Optimal sets: the trial points need to be rich enough

# Almost sure uniform convergence to a limit $V_t^*$

If the Bellman operators $\mathcal{B}_t$ are order-preserving "+" mild technical assumptions on $\mathcal{B}_t$ and the basic functions, we have

**Existence of an approximating limit**

Let $t \in [\![0, T]\!]$ be fixed. The sequence of functions $\left(V_t^k\right)_{k\in\mathbb{N}}$ generated by TDP $\mu$-a.s. converges uniformly on every compact set included in the domain of $V_t$ to a function $V_t^*$.

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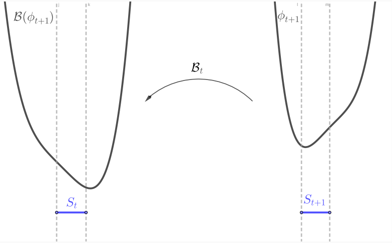
Is $V_t^*$ equal to $V_t$ ?

# Optimal sets: the trial points need to be rich enough

## Optimal sets

Let $(\phi_t)_{t\in[\![0,T]\!]}$ be $T+1$ functions. A sequence of sets $(S_t)_{t\in[\![0,T]\!]}$ is said to be $(\phi_t)$-optimal if for every $t \in [\![0, T-1]\!]$

$$\mathcal{B}_t\left(\phi_{t+1} + \delta_{S_{t+1}}\right) + \delta_{S_t} = \mathcal{B}_t\left(\phi_{t+1}\right) + \delta_{S_t}.$$



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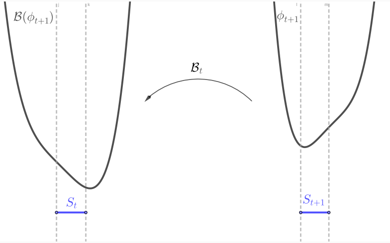


In order to compute $\mathcal{B}_t\left(\phi_{t+1}\right)$ restricted to $S_t$, one only needs to know $\phi_{t+1}$ restricted to $S_{t+1}$.

# $V_t^*$ is almost surely equal to $V_t$ on a set of interest

Almost surely, the approximations $\left(V_t^k\right)_k$ converges uniformly to $V_t^*$, which is equal to $V_t$ on a set of interest

**Convergence of TDP [Akian, Chancelier, T., 2018]**

Define $K_t^* := \limsup_k \operatorname{supp}\left(\mu_t^k\right)$, for every time $t \in [\![0, T]\!]$.
Assume that, $\mu$-a.s the sets $(K_t^*)_{t \in [\![0,T]\!]}$ are

- $(V_t)$-optimal if $\mathsf{opt} = \inf$,
- $(V_t^*)$-optimal if $\mathsf{opt} = \sup$.

Then, $\mu$-a.s. for every $t \in [\![0, T]\!]$ the function $V_t^*$ is equal to the value function $V_t$ on $K_t^*$.

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Then, $\mu$-a.s. for every $t \in [\![0, T]\!]$ the function $V_t^*$ is equal to the value function $V_t$ on $K_t^*$.

This is the usual convergence result for SDDP, new for a Min-Plus method

# Rough scheme of the proof, details in [Akian, Chancelier, T., 2018]

- $(V_t^k)_k$ converges uniformly to $V_t^*$ on every compact in the domain of $V_t$ by Arzela-Ascoli theorem

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- $\left(V_t^k\right)_k$ converges uniformly to $V_t^*$ on every compact in the domain of $V_t$ by Arzela-Ascoli theorem
- $(V_t^*)_t$ satisfies a system of **restricted** Bellman Equations on the sets $(K_t^*)$:

$$\begin{cases} V_T^* + \delta_{K_T^*} = \psi + \delta_{K_T^*} \\ \forall t \in [\![0, T-1]\!],\ \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{K_t^*} = V_t^* + \delta_{K_t^*} \end{cases} \tag{1}$$

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- If the sets $(K_t^*)_t$ are $(V_t^*)$-optimal when $\mathbf{opt} = \mathbf{sup}$ [1], satisfying (1) is enough to ensure that $V_t^* = V_t$ over $K_t^*$

[1] resp. $(V_t)$-optimality of $(K_t^*)_t$ when $\mathbf{opt} = \mathbf{inf}$

# Deterministic linear-quadratic optimal control with one constrained control

Let $\beta, \gamma$ be such that $\beta < \gamma$, we study the following Multistage convex optimization problem involving a constraint on one of the controls denoted by $v$:

$$\min_{\substack{x=(x_0,\dots,x_T)\\ u=(u_0,\dots,u_{T-1})\\ v=(v_0,\dots,v_{T-1})}} \sum_{t=0}^{T-1} c_t(x_t, u_t, v_t) + \psi(x_T)$$

$$\text{s.t.} \begin{cases} x_0 \in \mathbb{X} \text{ is given}, \\ \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t(x_t, u_t, v_t) \\ \forall t \in [\![0, T-1]\!], (u_t, v_t) \in \mathbb{U} \times [\beta, \gamma], \end{cases}$$

where $f_t$ is linear, $c_t$ and $\psi$ are convex quadratic.

# SDDP selection function: Quadratic Programming

**SDDP selection function, through a QP**

$$b = \min_{\substack{x' \in X \\ (u,v) \in \mathbb{U} \times [\beta,\gamma] \\ \lambda \in \mathbb{R}}} \left[ c_t\left(x', u, v\right) + \lambda \right]$$

$$\text{s.t.} \begin{cases} x' = x \\ \phi\left(f_t\left(x', u, v\right)\right) \leq \lambda \quad \forall \phi \in \Phi \end{cases} .$$

Denote by $b$ its optimal value and by $a$ a Lagrange multiplier of the constraint $x' - x = 0$ at the optimum

$$\phi_t^{\mathrm{SDDP}}\left(\Phi, x\right) := x' \mapsto \langle a, x' - x \rangle + b \ .$$

# Min-Plus selection function: closed form formula

Discretize the constrained control: $v \in \mathbb{V}_N$ where $\mathbb{V}_N \subset \mathbb{V}$ is of cardinal $N \in \mathbb{N}$. Add one dimension $y \in \mathbb{R}$ at the state space and homogeneize the costs and dynamics.

Details in the CDC paper.

**Min-Plus selection function**

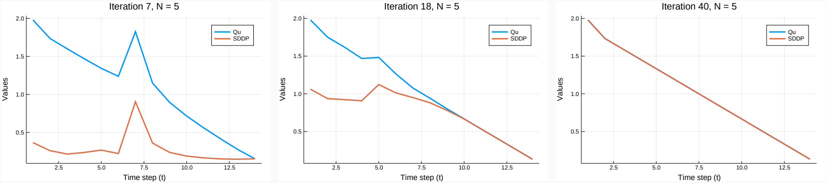
$$\phi_t^{\text{min-plus}}(\Phi, x, y) = \mathcal{B}_t^v(\phi)$$

$$\text{for some} \quad (v, \phi) \in \underbrace{\operatorname*{arg\,min}_{(v,\phi)\in\mathbb{V}_N\times\Phi} \mathcal{B}_t^v(\phi)}_{\substack{\text{Best image of current approximation} \\ \text{at the trial point}}} \underbrace{(x, y)}_{\text{trial point}} .$$

$\mathcal{B}_t^v(\phi)$ is given by a closed form formula (Discrete Algebraic Riccati formula).

# Numerical illustration on a toy example: converging gap

The gap between upper and lower approximations converges to 0 along the current optimal trajectories of SDDP




- Plots of $\overline{V}_t^k(x_t^k)$ and $\underline{V}_t^k(x_t^k)$ with $t$ in abscisses
- After 7 iterations (left), 18 iterations (middle) and 40 iterations (right)
- It is not straightforward to use a Min-Plus algorithm here, see the CDC paper for details.

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- Basic functions added at each step have to be tight and valid
- Trial points have to be "rich enough": either $V_t$-optimal (for upper approximations) or $V_t^*$-optimal (for lower approximations) is sufficent
- Upcoming: this work can easily be extended to a stochastic framework with white finite noises. Also, upper and lower bounds can be refined on a same set of points.

# References


- Marianne Akian, Jean-Philippe Chancelier, and Benoît Tran.
  A stochastic algorithm for deterministic multistage optimization problems.
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- Marianne Akian, Jean-Philipe Chancelier, and Benoît Tran.
  A min-plus-sddp algorithm for deterministic multistage convex programming.
  In *58th IEEE Conference on Decision and Control*, 2019.



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Thank you !

# Additional notations

- opt an operation that is either the pointwise infimum or the pointwise supremum of functions.
- $\overline{\mathbb{R}}$ the extended reals endowed with the operations $+\infty + (-\infty) = -\infty + \infty = +\infty$.
- For every $t \in [\![0, T]\!]$, fix $\mathsf{F}_t$ and $\mathbb{F}_t$ two subsets of $\left(\overline{\mathbb{R}}\right)^{\mathbb{X}}$ the set of functions on $\mathbb{X}$ such that $\mathsf{F}_t \subset \mathbb{F}_t$.
- A function $\phi$ is a basic function if $\phi \in \mathsf{F}_t$ for some $t \in [\![0, T]\!]$.
- For every set $X \subset \mathbb{X}$, denote by $\delta_X$ the function equal to 0 on $X$ and $+\infty$ elsewhere.
- For every $t \in [\![0, T]\!]$ and every set of basic functions $\Phi_t \subset \mathsf{F}_t$, we denote by $\mathcal{V}_{\Phi_t}$ its pointwise optimum, $\mathcal{V}_{\Phi_t} := \mathrm{opt}_{\phi \in \Phi_t} \phi$, *i.e.*

$$\begin{aligned} \mathcal{V}_{\Phi_t} : \quad \mathbb{X} &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto \mathrm{opt}\left\{\phi(x) \mid \phi \in \Phi_t\right\}. \end{aligned} \tag{2}$$

# Structural assumptions i

- **Common regularity:** for every $t \in [\![0, T]\!]$, there exists a common (local) modulus of continuity of all $\phi \in \mathbb{F}_t$.
- **Final condition:** for some $\Phi_T$ of $\mathsf{F}_T$, $\psi := \mathcal{V}_{\Phi_T}$.
- **Stability by the Bellman operators:** if $\phi \in \mathbb{F}_{t+1}$, then $\mathcal{B}_t(\phi)$ belongs to $\mathbb{F}_t$.
- **Stability by pointwise optimum:** if $\Phi_t \subset \mathsf{F}_t$ then $\mathcal{V}_{\Phi_t} \in \mathbb{F}_t$.
- **Stability by pointwise convergence:** if $\left(\phi^k\right)_{k\in\mathbb{N}} \subset \mathbb{F}_t$ converges pointwise to $\phi$ on the domain of $V_t$, then $\phi \in \mathbb{F}_t$.
- **Order preserving operators:** $\phi \leq \varphi$ implies $\mathcal{B}_t(\phi) \leq \mathcal{B}_t(\varphi)$.
- **Existence of the value functions:** the solution $(V_t)_{t\in[\![0,T]\!]}$ exist and each $V_t$ is proper.

# Structural assumptions ii

- **Existence of optimal sets:** for every compact set $K_t \subset \mathrm{dom}\,(V_t)$, for every function $\phi \in \mathbb{F}_{t+1}$ and constant $\lambda \in \mathbb{R}$, there exists a compact set $K_{t+1} \subset \mathrm{dom}\,(V_{t+1})$ such that we have

$$\mathcal{B}_t \left(\phi + \lambda + \delta_{K_{t+1}}\right) \leq \mathcal{B}_t \left(\phi + \lambda\right) + \delta K_t.$$

- **Additively subhomogeneous operators:** for every compact set $K_t$, there exists $M_t > 0$ s.t. for every constant function $\lambda$ and every function $\phi \in \mathbb{F}_{t+1}$, we have

$$\mathcal{B}_t \left(\phi + \lambda\right) + \delta K_t \leq \mathcal{B}_t \left(\phi\right) + \lambda M_t + \delta K_t.$$

# Discretization of the constrained control

Fix an integer $N \geq 2$, set $v_i = \beta + i\frac{\gamma-\beta}{N-1}$ for every $0 \leq i \leq N-1$ and set $\mathbb{V} := \{v_0, v_1, \dots v_{N-1}\}$. We define the following unconstrained switched multistage linear quadratic problem:

$$\min_{\substack{x \in \mathbb{X}^T \\ (u,v) \in (\mathbb{U} \times \mathbb{V})^{T-1}}} \sum_{t=0}^{T-1} c_t^{v_t}(x_t, u_t) + \psi(x_T)$$

$$\text{s.t. } \begin{cases} x_0 \in \mathbb{X} \text{ is given,} \\ \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t^{v_t}(x_t, u_t) \\ \forall t \in [\![0, T-1]\!],\ v_t \in \mathbb{V}\ , \end{cases}$$

# Homogeneization

Define the homogeneized costs and dynamics

$$\tilde{f}_t^{\,v}(x, y, u) = \begin{pmatrix} A_t & vb_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} B_t \\ 0 \end{pmatrix} u,$$

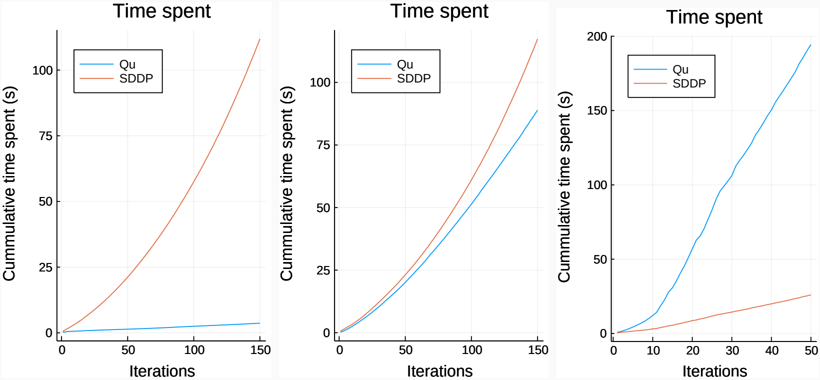
$$\tilde{c}_t^{\,v}(x, y, u) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} C_t & 0 \\ 0 & v^2 d_t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u^T D_t u,$$

Unconstrained 2-homogeneous MCP

$$\min_{\substack{(x,y)\in(\mathbb{X}\times\mathbb{R})^T \\ (u,v)\in(\mathbb{U}\times\mathbb{V})^{T-1}}} \sum_{t=0}^{T-1} \tilde{c}_t^{\,v_t}(x_t, y_t, u_t) + \widetilde{\psi}(x_T, y_T)$$

$$\text{s.t. } \begin{cases} (x_0, y_0) \in \mathbb{X} \times \mathbb{R} \text{ is given}, \\ \forall t \in [\![0, T-1]\!],\ (x_{t+1}, y_{t+1}) = \tilde{f}_t^{\,v_t}(x_t, y_t, u_t) \ . \end{cases}$$

# Numerical illustration on a toy example: time spent




- When the system does not often switch, the Min-plus part is fast. The more the system switches, the more the Min-plus algorithm is time consuming.
- SDDP is not affected by the switching aspect.

# Multistage Stochastic Convex Programming (MSCP)

MSCP can be solved by Dynamic Programming

$$\min_{(\mathsf{X},\mathsf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t\left(\mathsf{X_t},\mathsf{U_t},\mathsf{W_{t+1}}\right) + \psi\left(\mathsf{X_T}\right)\right]$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!]$$

$$\mathsf{X_{t+1}} = f_t\left(\mathsf{X_t},\mathsf{U_t},\mathsf{W_{t+1}}\right), \mathsf{X_0} \text{ given}$$

$$\sigma\left(\mathsf{U_t}\right) \subset \sigma\left(\mathsf{W_0},\dots,\mathsf{W_{t+1}}\right)$$

where the noise process $(\mathsf{W_t})_{t\in[\![1,T]\!]}$ is an independent sequence of random variables of finite supports

$$\tilde{\mathcal{B}}_t\left(\phi\right)\left(x,w\right) = \min_u c_t\left(x,u,w\right) + \phi\left(f_t\left(x,u,w\right)\right)$$

$$\mathcal{B}_t\left(\phi\right)\left(x\right) = \mathbb{E}\left[\tilde{\mathcal{B}}_t\left(x,\mathsf{W_{t+1}}\right)\right]$$

# Current optimal trajectories of Baucke-Downward-Zackeri

**Input:** $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

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Forward in time

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- Set $x_{t+1}^k := f_t\left(x_t^k, u_t^k\left(w_t^k\right), w_t^k\right)$ and iterate

# Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t\in[\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

Backward in time

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Backward in time

- Compute a new upper basic function $\overline{\phi}$ by evaluating a selection function at $\overline{\Phi}_{t+1}^{k+1}$ and $x_t^k$

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