A Min-Plus / SDDP Algorithm for Multistage Stochastic Convex Programming

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September 19th 2019

FGS 2019

Dynamic Programming and Bellman operators

Given an integer $T > 0$, consider the Dynamic Programming equations

$$\begin{cases} V_T = \psi \\ \forall t \in [\![0, T-1]\!],\ V_t = \mathcal{B}_t(V_{t+1}) \end{cases}$$

where

- Ψ is a function called the final cost function
- $\mathcal{B}_t$ is an operator called the Bellman operator
- V_t is called the value function at time $t \in [\![0, T]\!]$
- We want to compute $V_0(x_0)$ at some given state x_0

Multistage Stochastic (Convex) Programming (MSP)

Consider the MSP

$$\min_{(\mathsf{X},\mathsf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t\left(\mathsf{X}_\mathsf{t}, \mathsf{U}_\mathsf{t}, \mathsf{W}_{\mathsf{t+1}}\right) + \psi\left(\mathsf{X}_\mathsf{T}\right)\right]$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!]$$

$$\mathsf{X}_{\mathsf{t+1}} = f_t\left(\mathsf{X}_\mathsf{t}, \mathsf{U}_\mathsf{t}, \mathsf{W}_{\mathsf{t+1}}\right), \mathsf{X}_0 \text{ given}$$

$$\sigma\left(\mathsf{U}_\mathsf{t}\right) \subset \sigma\left(\mathsf{W}_1, \dots, \mathsf{W}_{\mathsf{t+1}}\right)$$

where the noise process $(\mathsf{W}_\mathsf{t})_{t\in[\![1,T]\!]}$ is an independent sequence of random variables of finite supports.

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Consider the MSP

$$\min_{(X,U)} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t\left(X_t, U_t, W_{t+1}\right) + \psi\left(X_T\right)\right]$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!]$$

$$X_{t+1} = f_t\left(X_t, U_t, W_{t+1}\right), X_0 \text{ given}$$

$$\sigma\left(U_t\right) \subset \sigma\left(W_1, \dots, W_{t+1}\right)$$

where the noise process $(W_t)_{t\in[\![1,T]\!]}$ is an independent sequence of random variables of finite supports.

MSP can be solved by Dynamic Programming by setting
$\tilde{\mathcal{B}}_t\left(\phi\right)\left(x, w\right) = \min_u c_t\left(x, u, w\right) + \phi\left(f_t\left(x, u, w\right)\right)$
$\mathcal{B}_t\left(\phi\right)\left(x\right) = \mathbb{E}\left[\tilde{\mathcal{B}}_t\left(x, W_{t+1}\right)\right]$

What we will do

Build an algorithm that builds approximations of the value functions V_t based on properties of the Bellman operators $\mathcal{B}_t$, e.g. monotonicity

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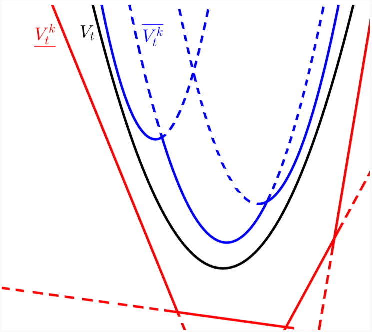
It must generalize the Stochastic Dual Dynamic Programming (SDDP) algorithm (developed by Pereira and Pinto 1991, Shapiro 2011, ...)
and the Min-plus algorithm for deterministic control problems (developed by McEneaney 2007, Qu 2014)

Overview of our algorithm



Lower approximations $\underline{V_t}^k$ as a supremum of basic functions (affine functions for SDDP) below V_t

Upper approximations $\overline{V_t}^k$ as an infimum of some **other** basic functions (quadratic functions for Min-Plus) above V_t

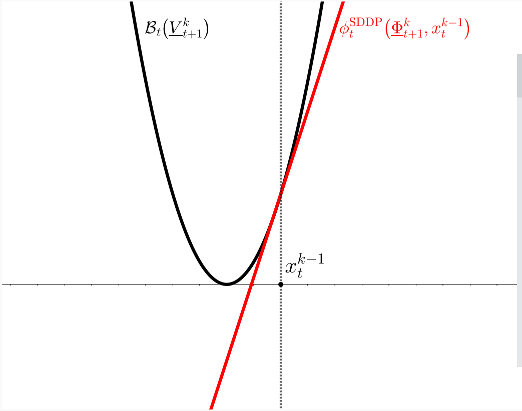
Outline

1. Tropical Dynamic Programming (TDP): an algorithm encompassing both SDDP and a Min-Plus algorithm

2. Convergence result of TDP

3. Converging upper and lower approximations for Multistage Stochastic (Convex) Programming

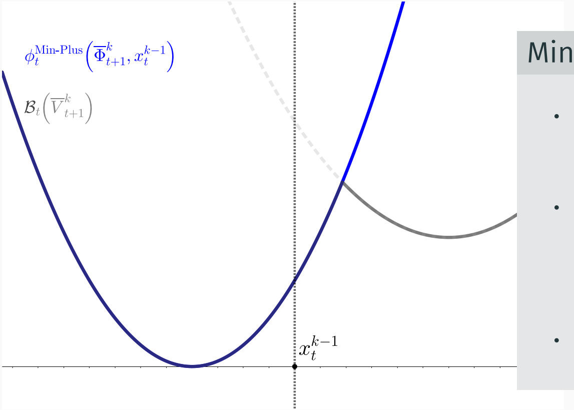
Trial points and selection functions: SDDP exemple



SDDP Exemple

- Affine functions
- Lower approximations opt $=$ sup
- $\underline{V}_{t+1}^k := \sup_{\phi \in \underline{\Phi}_{t+1}^k} \phi$

Trial points and selection functions: Min-Plus exemple



Min-Plus Exemple

- Quadratic functions
- Upper approximations opt $=$ inf
- $\overline{V}_{t+1}^k := \inf_{\phi \in \overline{\Phi}_{t+1}^k} \phi$

Tight and Valid selection functions

Tightness Assumption

$$\underbrace{\overbrace{\phi_t^{\mathrm{SDDP}}}^{\text{Selection function}}\left(\overbrace{\underline{\Phi}_{t+1}^k}^{\text{Set of basic functions}}, x_t^{k-1}\right)}_{\text{Basic function}}\big(\overbrace{x_t^{k-1}}^{\text{Trial point}}\big) = \mathcal{B}_t\left(\underline{V}_{t+1}^k\right)\left(x_t^{k-1}\right)$$

It is a local property.

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Validity Assumption

$$\phi_t^{\mathrm{SDDP}}\left(\underline{\Phi}_{t+1}^{k}, x_t^{k-1}\right) \leq \mathcal{B}_t\left(\underline{V}_{t+1}^{k}\right) \quad (\mathrm{SDDP}) \quad \textbf{opt} = \textbf{sup}$$

$$\phi_t^{\text{Min-Plus}}\left(\overline{\Phi}_{t+1}^{k}, x_t^{k-1}\right) \geq \mathcal{B}_t\left(\overline{V}_{t+1}^{k}\right) \quad (\text{Min-Plus}) \quad \textbf{opt} = \textbf{inf}$$

It is a global property.

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1. Initialize the approximations to infinity.

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$$\Phi_t^{k+1} = \Phi_t^k \cup \{\phi\} \,.$$

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5. **Update:** knowing the updated set of approximations $\left(\Phi_t^{k+1}\right)_t$ compute a new probability law μ^{k+1}.

2. Convergence result of TDP

Almost sure uniform convergence to a limit V_t^*

If the Bellman operators $\mathcal{B}_t$ are order-preserving "+" mild technical assumptions on $\mathcal{B}_t$ and the basic functions, we have

Existence of an approximating limit

Let $t \in [\![0, T]\!]$ be fixed. The sequence of functions $\left(V_t^k\right)_{k\in\mathbb{N}}$ generated by TDP μ-a.s. converges uniformly on every compact set included in the domain of V_t to a function V_t^*.

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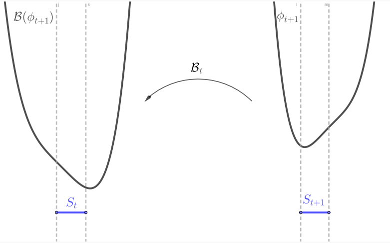
Is V_t^* equal to V_t ?

Optimal sets: the trial points need to be rich enough

Optimal sets

Let $(\phi_t)_{t\in[\![0,T]\!]}$ be $T+1$ functions. A sequence of sets $(S_t)_{t\in[\![0,T]\!]}$ is said to be (ϕ_t)-optimal if for every $t \in [\![0, T-1]\!]$

$$\mathcal{B}_t\left(\phi_{t+1} + \delta_{S_{t+1}}\right) + \delta_{S_t} = \mathcal{B}_t\left(\phi_{t+1}\right) + \delta_{S_t}.$$



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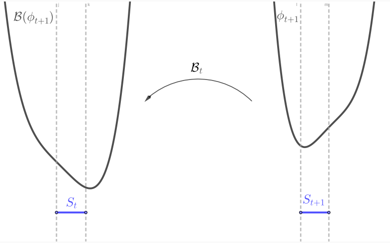

In order to compute $\mathcal{B}_t\left(\phi_{t+1}\right)$ restricted to S_t, one only needs to know ϕ_{t+1} restricted to S_{t+1}.

V_t^* is almost surely equal to V_t on a set of interest

Almost surely, the approximations $\left(V_t^k\right)_k$ converges uniformly to V_t^*, which is equal to V_t on a set of interest

Convergence of TDP [Akian, Chancelier, T., 2018]

Define $K_t^* := \limsup_k \operatorname{supp}\left(\mu_t^k\right)$, for every time $t \in [\![0, T]\!]$.
Assume that, μ-a.s the sets $(K_t^*)_{t \in [\![0,T]\!]}$ are

- (V_t)-optimal if $\mathsf{opt} = \inf$,
- (V_t^*)-optimal if $\mathsf{opt} = \sup$.

Then, μ-a.s. for every $t \in [\![0, T]\!]$ the function V_t^* is equal to the value function V_t on K_t^*.

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Then, μ-a.s. for every $t \in [\![0, T]\!]$ the function V_t^* is equal to the value function V_t on K_t^*.

This is the usual convergence result for SDDP, new for a Min-Plus method

- $\left(V_t^k\right)_k$ converges uniformly to V_t^* on every compact in the domain of V_t by Arzela-Ascoli theorem

[1] resp. (V_t)-optimality of $(K_t^*)_t$ when $\mathbf{opt} = \mathbf{inf}$

Rough scheme of the proof, details in [Akian, Chancelier, T., 2018]

- $(V_t^k)_k$ converges uniformly to V_t^* on every compact in the domain of V_t by Arzela-Ascoli theorem
- $(V_t^*)_t$ satisfies a system of **restricted** Bellman Equations on the sets (K_t^*):

$$\begin{cases} V_T^* + \delta_{K_T^*} = \psi + \delta_{K_T^*} \\ \forall t \in [\![0, T-1]\!],\ \mathcal{B}_t\left(V_{t+1}^*\right) + \delta_{K_t^*} = V_t^* + \delta_{K_t^*} \end{cases} \tag{1}$$

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- If the sets $(K_t^*)_t$ are (V_t^*)-optimal when $\mathbf{opt} = \mathbf{sup}$ [1], satisfying (1) is enough to ensure that $V_t^* = V_t$ over K_t^*

[1] resp. (V_t)-optimality of $(K_t^*)_t$ when $\mathbf{opt} = \mathbf{inf}$

Deterministic linear-quadratic optimal control with one constrained control

Let β, γ be such that $\beta < \gamma$, we study the following Multistage convex optimization problem involving a constraint on one of the controls denoted by v:

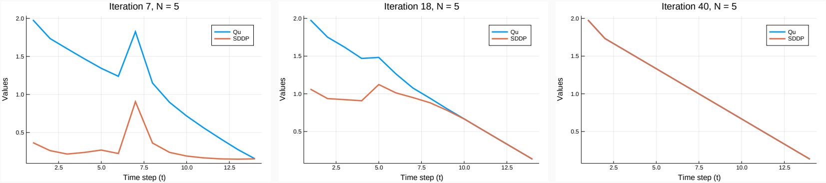
$$\min_{\substack{x=(x_0,\dots,x_T)\\ u=(u_0,\dots,u_{T-1})\\ v=(v_0,\dots,v_{T-1})}} \sum_{t=0}^{T-1} c_t(x_t, u_t, v_t) + \psi(x_T)$$

$$\text{s.t. } \begin{cases} x_0 \in \mathbb{X} \text{ is given,} \\ \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t(x_t, u_t, v_t) \\ \forall t \in [\![0, T-1]\!], (u_t, v_t) \in \mathbb{U} \times [\beta, \gamma], \end{cases}$$

where f_t is linear, c_t and ψ are convex quadratic.

Numerical illustration on a toy example: converging gap

The gap between upper and lower approximations converges to 0 along the current optimal trajectories of SDDP



- Plots of $\overline{V}_t^k(x_t^k)$ and $\underline{V}_t^k(x_t^k)$ with t in abscisses
- After 7 iterations (left), 18 iterations (middle) and 40 iterations (right)
- It is not straightforward to use a Min-Plus algorithm here, see [Akian, Chancelier, T., CDC 2019]

3. Converging upper and lower approximations for Multistage Stochastic (Convex) Programming

3.1 Upper and lower approximations may converge on different points

3.2 Converging upper and lower approximations along current optimal trajectories

Upper and lower approximations may converge on different points

We can either build upper approximations or lower approximations using TDP but…

Upper and lower approximations may converge on different points

How to make upper and lower approximations converge on the same points

Converging upper and lower approximations along current optimal trajectories

- In MSPs with finite noise build a worst case deterministic optimal trajectory for the lower approximations ("Problem-Child" method of Baucke, Downward and Zackeri 2018) from a deterministic criterium

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Converging upper and lower approximations along current optimal trajectories. (Work in progress)

On every accumulation point x_t^* of the deterministic current optimal trajectories (x_t^k) we have that

$$\underline{V}_t^k(x_t^*) = V_t(x_t^*) = \overline{V}_t^k(x_t^*)$$

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- Convergence result identical to the SDDP literature, new for the Min-Plus method (and maybe others)
- Basic functions added at each step have to be tight and valid
- Trial points have to be "rich enough": either V_t-optimal (for upper approximations) or V_t^*-optimal (for lower approximations) is sufficent
- One can use the optimal trajectories of lower approximations (SDDP) in order to build upper approximations (Min-Plus) and get exact converging upper and lower bounds

Marianne Akian, Jean-Philippe Chancelier, and Benoît Tran.
A stochastic algorithm for deterministic multistage optimization problems.
arXiv:1810.12870 [math], October 2018.

Marianne Akian, Jean-Philipe Chancelier, and Benoît Tran.
A min-plus-sddp algorithm for deterministic multistage convex programming.
In *(To appear) 58th IEEE Conference on Decision and Control*, 2019.

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Thank you !

Additional notations

- opt an operation that is either the pointwise infimum or the pointwise supremum of functions.
- $\overline{\mathbb{R}}$ the extended reals endowed with the operations $+\infty + (-\infty) = -\infty + \infty = +\infty$.
- For every $t \in [\![0, T]\!]$, fix F_t and $\mathbb{F}_t$ two subsets of $\left(\overline{\mathbb{R}}\right)^{\mathbb{X}}$ the set of functions on $\mathbb{X}$ such that $\mathsf{F}_t \subset \mathbb{F}_t$.
- A function ϕ is a basic function if $\phi \in \mathsf{F}_t$ for some $t \in [\![0, T]\!]$.
- For every set $X \subset \mathbb{X}$, denote by δ_X the function equal to 0 on X and $+\infty$ elsewhere.
- For every $t \in [\![0, T]\!]$ and every set of basic functions $\Phi_t \subset \mathsf{F}_t$, we denote by $\mathcal{V}_{\Phi_t}$ its pointwise optimum, $\mathcal{V}_{\Phi_t} := \mathrm{opt}_{\phi \in \Phi_t} \phi$, *i.e.*

$$\begin{aligned} \mathcal{V}_{\Phi_t} : \quad \mathbb{X} &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto \mathrm{opt}\left\{\phi(x) \mid \phi \in \Phi_t\right\}. \end{aligned} \tag{2}$$

Structural assumptions i

- **Common regularity:** for every $t \in [\![0, T]\!]$, there exists a common (local) modulus of continuity of all $\phi \in \mathbb{F}_t$.
- **Final condition:** for some Φ_T of F_T, $\psi := \mathcal{V}_{\Phi_T}$.
- **Stability by the Bellman operators:** if $\phi \in \mathbb{F}_{t+1}$, then $\mathcal{B}_t(\phi)$ belongs to $\mathbb{F}_t$.
- **Stability by pointwise optimum:** if $\Phi_t \subset \mathsf{F}_t$ then $\mathcal{V}_{\Phi_t} \in \mathbb{F}_t$.
- **Stability by pointwise convergence:** if $\left(\phi^k\right)_{k \in \mathbb{N}} \subset \mathbb{F}_t$ converges pointwise to ϕ on the domain of V_t, then $\phi \in \mathbb{F}_t$.
- **Order preserving operators:** $\phi \leq \varphi$ implies $\mathcal{B}_t(\phi) \leq \mathcal{B}_t(\varphi)$.
- **Existence of the value functions:** the solution $(V_t)_{t \in [\![0,T]\!]}$ exist and each V_t is proper.

Structural assumptions ii

- **Existence of optimal sets:** for every compact set $K_t \subset \operatorname{dom}(V_t)$, for every function $\phi \in \mathbb{F}_{t+1}$ and constant $\lambda \in \mathbb{R}$, there exists a compact set $K_{t+1} \subset \operatorname{dom}(V_{t+1})$ such that we have

$$\mathcal{B}_t\left(\phi + \lambda + \delta_{K_{t+1}}\right) \leq \mathcal{B}_t\left(\phi + \lambda\right) + \delta K_t.$$

- **Additively subhomogeneous operators:** for every compact set K_t, there exists $M_t > 0$ s.t. for every constant function λ and every function $\phi \in \mathbb{F}_{t+1}$, we have

$$\mathcal{B}_t\left(\phi + \lambda\right) + \delta K_t \leq \mathcal{B}_t\left(\phi\right) + \lambda M_t + \delta K_t.$$

SDDP selection function

We define SDDP selection function through the following QP

$$b = \min_{\substack{x' \in X \\ (u,v) \in \mathbb{U} \times [\beta,\gamma] \\ \lambda \in \mathbb{R}}} \left[c_t \left(x', u, v \right) + \lambda \right]$$

$$\text{s.t.} \begin{cases} x' = x \\ \phi \left(f_t \left(x', u, v \right) \right) \leq \lambda \quad \forall \phi \in \mathbf{\Phi} \end{cases} .$$

Denote by b its optimal value and by a a Lagrange multiplier of the constraint $x' - x = 0$ at the optimum

$$\phi_t^{\mathrm{SDDP}} \left(\mathbf{\Phi}, x \right) := x' \mapsto \langle a, x' - x \rangle + b \ .$$

Finally, at time $t = T$, for any $\mathbf{\Phi} \subset \mathrm{F}_T^{\mathrm{SDDP}}$ and $x \in \mathbb{X}$, fix $a \in \partial V_T(x)$ and define

$$\phi_T^{\mathrm{SDDP}} \left(\mathbf{\Phi}, x \right) := x' \mapsto \langle a, x' - x \rangle + V_T \left(x \right) .$$

Discretization of the constrained control

Fix an integer $N \geq 2$, set $v_i = \beta + i\frac{\gamma-\beta}{N-1}$ for every $0 \leq i \leq N-1$ and set $\mathbb{V} := \{v_0, v_1, \dots v_{N-1}\}$. We define the following unconstrained switched multistage linear quadratic problem:

$$\min_{\substack{x \in \mathbb{X}^T \\ (u,v) \in (\mathbb{U} \times \mathbb{V})^{T-1}}} \sum_{t=0}^{T-1} c_t^{v_t}(x_t, u_t) + \psi(x_T)$$

$$\text{s.t.} \begin{cases} x_0 \in \mathbb{X} \text{ is given,} \\ \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t^{v_t}(x_t, u_t) \\ \forall t \in [\![0, T-1]\!],\ v_t \in \mathbb{V}\ , \end{cases}$$

Homogeneization

Define the homogeneized costs and dynamics

$$\tilde{f}_t^{\,v}(x, y, u) = \begin{pmatrix} A_t & vb_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} B_t \\ 0 \end{pmatrix} u,$$

$$\tilde{c}_t^{\,v}(x, y, u) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} C_t & 0 \\ 0 & v^2 d_t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u^T D_t u,$$

Unconstrained 2-homogeneous MCP

$$\min_{\substack{(x,y)\in(\mathbb{X}\times\mathbb{R})^T \\ (u,v)\in(\mathbb{U}\times\mathbb{V})^{T-1}}} \sum_{t=0}^{T-1} \tilde{c}_t^{\,v_t}(x_t, y_t, u_t) + \widetilde{\psi}(x_T, y_T)$$

$$\text{s.t.} \begin{cases} (x_0, y_0) \in \mathbb{X} \times \mathbb{R} \text{ is given,} \\ \forall t \in [\![0, T-1]\!], \ (x_{t+1}, y_{t+1}) = \tilde{f}_t^{\,v_t}(x_t, y_t, u_t) \ . \end{cases}$$

Min-Plus selection function

We define the selection function $\phi_t^{\text{min-plus}}$ as follows. For any given $\Phi \subset \mathsf{F}_{t+1}^{\text{min-plus}}$ and $(x,y) \in \mathbb{X} \times \mathbb{R}$,

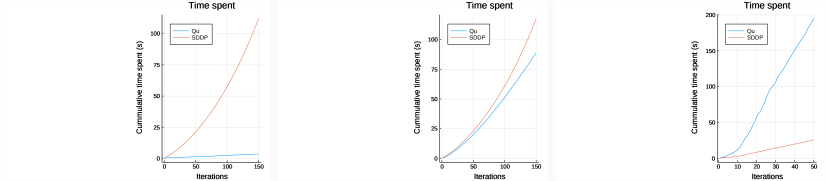
$$\phi_t^{\text{min-plus}}(\Phi, x, y) = \mathcal{B}_t^v(\phi)$$

$$\text{for some} \quad (v,\phi) \in \underbrace{\operatorname*{arg\,min}_{(v,\phi)\in\mathbb{V}\times\Phi} \mathcal{B}_t^v(\phi)}_{\text{Best image of current approximation at trial point}} \underbrace{(x,y)}_{\text{trial point}} .$$

Moreover, at time $t = T$, for any $\Phi \subset \mathsf{F}_T^{\text{min-plus}}$ and $(x,y) \in \mathbb{X} \times \mathbb{R}$, we set

$$\phi_T^{\text{min-plus}}(\Phi, x, y) = \tilde{\psi}(x,y) = \psi(x).$$

Numerical results on a toy example: time spent



Time spent for the first example (left) and the second example when $N = 50$ (middle) and $N = 200$ (right).

Multistage Stochastic Convex Programming (MSCP)

MSCP can be solved by Dynamic Programming

$$\min_{(\mathsf{X},\mathsf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t\left(\mathsf{X_t}, \mathsf{U_t}, \mathsf{W_{t+1}}\right) + \psi\left(\mathsf{X_T}\right)\right]$$

$$\text{s.t. } \forall t \in [\![0, T-1]\!]$$

$$\mathsf{X_{t+1}} = f_t\left(\mathsf{X_t}, \mathsf{U_t}, \mathsf{W_{t+1}}\right), \mathsf{X_0} \text{ given}$$

$$\sigma\left(\mathsf{U_t}\right) \subset \sigma\left(\mathsf{W_0}, \dots, \mathsf{W_{t+1}}\right)$$

where the noise process $(\mathsf{W_t})_{t\in[\![1,T]\!]}$ is an independent sequence of random variables of finite supports

$\tilde{\mathcal{B}}_t\left(\phi\right)\left(x, w\right) = \min_u c_t\left(x, u, w\right) + \phi\left(f_t\left(x, u, w\right)\right)$

$\mathcal{B}_t\left(\phi\right)\left(x\right) = \mathbb{E}\left[\tilde{\mathcal{B}}_t\left(x, \mathsf{W_{t+1}}\right)\right]$

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

We construct a deterministic trajectory $\left(x_t^k\right)_{t\in[\![0,T]\!]}$, optimal (in the sense introduced beforehand) for the current approximations.

Forward in time

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

We construct a deterministic trajectory $\left(x_t^k\right)_{t \in [\![0,T]\!]}$, optimal (in the sense introduced beforehand) for the current approximations.

Forward in time

- Set $x_0^k := x_0$

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

We construct a deterministic trajectory $\left(x_t^k\right)_{t \in [\![0,T]\!]}$, optimal (in the sense introduced beforehand) for the current approximations.

Forward in time

- Set $x_0^k := x_0$
- For each noise w, compute an optimal control $u_t^k(w)$ to apply at x_t^k for the lower current approximation $\underline{V}_t^k$

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

We construct a deterministic trajectory $\left(x_t^k\right)_{t\in[\![0,T]\!]}$, optimal (in the sense introduced beforehand) for the current approximations.

Forward in time

- Set $x_0^k := x_0$
- For each noise w, compute an optimal control $u_t^k(w)$ to apply at x_t^k for the lower current approximation $\underline{V}_t^k$
- Find a noise w_{t+1}^k which maximises $\arg\max_w \left(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\right)\left(f_t\left(x_t^k, u_t^k(w), w\right)\right)$

Current optimal trajectories of Baucke-Downward-Zackeri

Input: $\left(\overline{V}_t^k\right)_t$ and $\left(\underline{V}_t^k\right)_t$ upper and lower current approximations generated by TDP given a Multistage stochastic convex optimization problem

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- Find a noise w_{t+1}^k which maximises $\arg\max_w \left(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\right)\left(f_t\left(x_t^k, u_t^k(w), w\right)\right)$
- Set $x_{t+1}^k := f_t\left(x_t^k, u_t^k\left(w_t^k\right), w_t^k\right)$ and iterate

Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t \in [\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

Backward in time

Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t \in [\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

Backward in time

- Compute a new upper basic function $\overline{\phi}$ by evaluating a selection function at $\overline{\Phi}_{t+1}^{k+1}$ and x_t^k

Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t \in [\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

Backward in time

- Compute a new upper basic function $\overline{\phi}$ by evaluating a selection function at $\overline{\Phi}_{t+1}^{k+1}$ and x_t^k
- Add the new basic function $\underline{\phi}$ to the current collection of upper basic functions $\overline{\Phi}_t^{k+1} = \overline{\Phi}_t^k \cup \left\{\overline{\phi}\right\}$

Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t \in [\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

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- Compute a new lower basic function $\underline{\phi}$ by evaluating a selection function at $\underline{\Phi}_{t+1}^{k+1}$ and x_t^k

Using the optimal trajectories of lower approximations (SDDP) as trial points for upper approximations (Min-Plus)

Denote by $\left(x_t^k\right)_{t\in[\![0,T]\!]}$ the deterministic current optimal trajectory of Baucke-Downward-Zackeri

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- Compute a new lower basic function $\underline{\phi}$ by evaluating a selection function at $\underline{\Phi}_{t+1}^{k+1}$ and x_t^k
- Add the new basic function $\underline{\phi}$ to the current collection of lower basic functions $\underline{\Phi}_t^{k+1} = \underline{\Phi}_t^k \cup \left\{\underline{\phi}\right\}$