Tropical Dynamic Programming for Lipschitz Multistage Stochastic Programming

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Multistage Stochastic Programming (MSP)

Consider the Multistage stochastic optimization problem

$$\min_{(X,U)} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t^{W_{t+1}}(X_t, U_t) + \psi(X_T)\right],$$

$$\text{s.t. } X_0 = x_0 \text{ given}, \forall t \in [\![0, T-1]\!],$$

$$X_{t+1} = f_t^{W_{t+1}}(X_t, U_t),$$

$$\sigma(U_t) \subset \sigma(W_1, \dots, W_{t+1}),$$

Assumption (Finite support independent noises)

The sequence $(W_t)_{t\in[\![1,T]\!]}$ is made of independent random variables each with finite support.

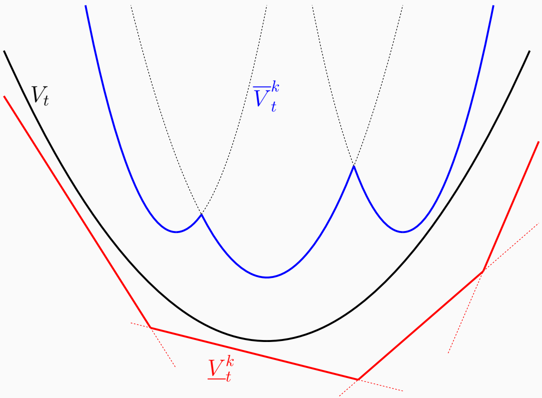
Bellman operators and Dynamic Programming

MSP can be solved by Dynamic Programming

- Pointwise Bellman operator:
  for all $w \in \mathrm{supp}(\mathsf{W}_{t+1})$ and $\phi : \mathbb{X} \to \overline{\mathbb{R}}$
  $$\mathcal{B}_t^w(\phi) : x \in \mathbb{X} \mapsto \min_u \Big(c_t^w(x,u) + \phi\big(f_t^w(x,u)\big) \Big) \in \overline{\mathbb{R}}$$
- (Average) Bellman operator:
  $$\mathfrak{B}_t(\phi) : x \in \mathbb{X} \mapsto \mathbb{E}\Big[\mathcal{B}_t^{\mathsf{W}_{t+1}}(\phi)(x)\Big] \in \overline{\mathbb{R}}$$
- Dynamic Programming Equations
  $$V_T = \psi \;\text{ and }\; \forall t \in [\![0, T-1]\!],\; V_t = \mathfrak{B}_t(V_{t+1})$$
    - V_t is called the value function at time $t \in [\![0, T]\!]$
- We want to compute $V_0(x_0)$ at some given state x_0

Goal: simultaenous Min-plus & Max-plus approximations of V_t

Build an algorithm that simultaneously generates upper and lower approximations of V_t as
min-plus linear and max-plus linear combinaisons of basic functions



Goal: simultaenous Min-plus & Max-plus approximations of V_t

Build an algorithm that simultaneously generates upper and lower approximations of V_t as min-plus linear and max-plus linear combinaisons of basic functions

- Generalizes the Min-plus algorithm for deterministic control problems (McEneaney 2007, Qu 2014) giving upper approximations as infima of quadratics
- and the Stochastic Dual Dynamic Programming (SDDP) algorithm (Pereira and Pinto 1991, Shapiro 2011, ...) giving lower approximations as suprema of affine cuts

Outline

1. Tropical Dynamic Programming (TDP): an algorithm building min-plus and max-plus approximations of value functions
2. Convergence result of TDP
3. Numerical example: linear-polyhedral framework

1. Tropical Dynamic Programming (TDP): an algorithm building min-plus and max-plus approximations of value functions

1.1 Lipschitz Multistage Stochastic optimization problems

1.2 How to select a new basic function ?

1.3 Problem-child trajectory of Baucke and al. (2018)

1.4 Tropical Dynamic Programming (TDP)

Lipschitz Multistage Stochastic optimization problems

Assumption (Lipschitz dynamic, costs and constraints)

For every time $t < T$ and $w \in \operatorname{supp}(\mathsf{W}_{t+1})$, the dynamics f_t^w, the costs c_t^w and the constraint set-valued mappings $\mathcal{U}_t^w$ are Lipschitz continuous on X_t, i.e. *for some constant $L_{\mathcal{U}_t^w} > 0$, for every $x_1, x_2 \in X_t$, $d_{\mathcal{H}}\left(\mathcal{U}_t^w(x_1), \mathcal{U}_t^w(x_2)\right) \leq L_{\mathcal{U}_t^w} \|x_1 - x_2\|$.*

Proposition (Lipschitz MSP implies regularity of $\mathfrak{B}_t$)

Let $\phi : \mathbb{X} \to \overline{\mathbb{R}}$ L_{t+1}-Lipschitz on X_{t+1} be given. The function $\mathfrak{B}_t(\phi)$ is L_t-Lipschitz on X_t for some constant $L_t > 0$ which only depends on the data of the MSP problem and L_{t+1}.

Constraint set-valued mapping

For each noise $w \in \mathrm{supp}(\mathsf{W}_{t+1})$, $t \in [\![0, T-1]\!]$, define the constraint set-valued mapping $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$

$$\mathcal{U}_t^w(x) := \left\{ u \in \mathbb{U} \mid c_t^w(x,u) < +\infty \text{ and } f_t^w(x,u) \in X_{t+1} \right\}.[1]$$

Assumption (Recourse assumption)

The set-valued mapping $\mathcal{U}_t^w$ is non-empty compact valued

Proposition (Known domains of V_t)

Under the recourse assumption, $\mathrm{dom}\, V_t = X_t$

[1] $\forall w \in \mathrm{supp}(\mathsf{W}_{t+1}), X_t^w := \pi_{\mathbb{X}}\left(\mathrm{dom}\, c_t^w\right)$, and $X_t := \cap_{w \in \mathrm{supp}(\mathsf{W}_{t+1})} X_t^w$.

How to select a new basic function ?

Given, x_t called trial point and $F_{t+1} \subset \mathsf{F}_{t+1}$ set of basic functions the selection function returns a function $\phi_t = S_t\left(F_{t+1}, x_t\right)$

Denote by $\mathcal{V}_{F_{t+1}}$ the sup or inf of basic functions in F_{t+1}

Tightness Assumption (local property)

$$\phi_t\left(x_t\right) = \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)\left(x_t\right)$$

Validity Assumption (global property)

$$\phi_t \leq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right) \quad \text{(Max-plus lin. combinaisons case)}$$

$$\phi_t \geq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right) \quad \text{(Min-plus lin. combinaisons case)}$$

Problem-child trajectory of Baucke and al. (2018)

Fix two sequences of functions $\underline{\phi}_0, \dots, \underline{\phi}_T$ and $\overline{\phi}_0, \dots, \overline{\phi}_T$

Recursively define a trajectory of states $x_0^*, \dots, x_T^*$ called the Problem-child trajectory. Initial state x_0^* is given, then **for** $t < T$

1. For all $w \in \mathrm{supp}\,(\mathbf{W}_{t+1})$, compute optimal control at x_t^*
$$u_t^w \in \operatorname*{arg\,min}_{u \in U} \left(c_t^w\,(x_t^*, u) + \underline{\phi}_{t+1}\big(f_t^w\,(x_t^*, u)\,\big) \right)$$
2. Compute "the worst" noise
$w^* \in \arg\max \big(\overline{\phi}_{t+1} - \underline{\phi}_{t+1}\big)\big(f_t^w\,(x_t^*, u_t^w)\,\big)$
3. Set $x_{t+1}^* = f_t^{w^*}\,\big(x_t^*, u_t^{w^*}\big)$

Interpretation

Problem child trajectory = "Worst" optimal trajectory of the lower approximations

Tropical Dynamic Programming (TDP) algorithm

Algorithm 1 Tropical Dynamic Programming (TDP)

Input: Compatible selection functions $(\overline{S}_t)_t$ and $(\underline{S}_t)_t$ and $(\mathrm{W_t})_{t \in [\![0, T-1]\!]}$ independent r.v. with finite support.

Output: Sequence of sets $\left(\overline{F}_t^k\right)_{k \in \mathbb{N}}$, $\left(\underline{F}_t^k\right)_{k \in \mathbb{N}}$ and associated functions $\overline{V}_t^k = \inf_{\phi \in \overline{F}_t^k} \phi$ and $\underline{V}_t^k = \sup_{\phi \in \underline{F}_t^k} \phi$

For every $t \in [\![0, T]\!]$, $\overline{F}_t^0 := \emptyset$ and $\underline{F}_t^0 := \emptyset$
for $k \geq 0$ **do**
    Forward. Compute Problem-child trajectory $\left(x_t^k\right)_{t \in [\![0, T]\!]}$
    for t from T to 0 **do**
        Backward. Set $\overline{\phi}_t := \overline{S}_t\left(\overline{F}_{t+1}^k, x_t^k\right)$ and $\underline{\phi}_t := \underline{S}_t\left(\underline{F}_{t+1}^k, x_t^k\right)$
        Add them, $\overline{F}_t^{k+1} := \overline{F}_t^k \cup \left\{\overline{\phi}_t\right\}$ and $\underline{F}_t^{k+1} := \underline{F}_t^k \cup \left\{\underline{\phi}_t\right\}$
    end for
end for

2. Convergence result of TDP

2.1 Uniform convergence to some limit functions

2.2 Asymptotic convergence of TDP

Convergence to limits $\underline{V}_t^*$ and $\overline{V}_t^*$

Under finite independent noises, Lipschitz data and recourse assumptions we have

Existence of an approximating limit

The sequence of functions $\left(\underline{V}_t^k\right)_{k\in\mathbb{N}}$ (resp. $\left(\overline{V}_t^k\right)_{k\in\mathbb{N}}$) generated by TDP converges uniformly on every compact set included in the domain of V_t to a function $\underline{V}_t^*$ (resp. $\overline{V}_t^*$).

Some features of TDP

- No need to discretize the state space
- $(\underline{V}_t^k)_k$ and $(\overline{V}_t^k)_k$ are monotonic
- $\underline{V}_t^*$ and $\overline{V}_t^*$ are close to V_t on “interesting points”, but may be far from V_t elsewhere.

Asymptotic convergence of TDP

Under finite independent noises, Lipschitz data and recourse assumptions we have

Convergence of TDP [Akian, Chancelier, T., 2020]

Denote by $\left(x_t^k\right)_{0\leq t\leq T}$ the k-th Problem-child trajectory.

For every accumulation point x_t^* of $\left(x_t^k\right)_{k\in\mathbb{N}}$, we have

$$\overline{V}_t^k\left(x_t^k\right) - \underline{V}_t^k\left(x_t^k\right) \underset{k\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \overline{V}_t^*\left(x_t^*\right) = V_t\left(x_t^*\right) = \underline{V}_t^*\left(x_t^*\right)$$

This result generalizes the convergence of SDDP à la [Philpott and al. (2013)] and [Baucke and al. (2018)] seen as a specific instance of TDP for the linear-polyhedral framework

Idea of the proof, details in [Akian, Chancelier, T., 2020]

- $\left(\underline{V}_t^k\right)_k$ (resp. $\left(\overline{V}_t^k\right)_k$) converges uniformly to $\underline{V}_t^*$ (resp. $\overline{V}_t^*$) on the domain of V_t by Arzela-Ascoli theorem
- Exploiting monotonicity of the approximations and that each operator $\mathcal{B}_t^w$ is order preserving

$$
\begin{aligned}
0 &\leq \overline{V}_t^{k+1}\left(x_t^k\right) - \underline{V}_t^{k+1}\left(x_t^k\right) \\
&\leq \sum_{w \in \mathrm{supp}(\mathbf{W}_{t+1})} \mathbb{P}\left[\mathbf{W}_{t+1} = w\right]\left[\left(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\right)\left(f_t\left(x_t^k, u_t^k(w), w\right)\right)\right]
\end{aligned}
$$

- Using that the PC-trajectory is the “worst” optimal trajectory then taking the limit in k

$$
0 \leq \overline{V}_t^*(x_t^*) - \underline{V}_t^*(x_t^*) \leq \overline{V}_{t+1}^*(x_{t+1}^*) - \underline{V}_{t+1}^*(x_{t+1}^*)
$$

- Conclude by backward recursion on t

Section content

3. Numerical example: linear-polyhedral framework

3.1 Linear-polyhedral framework

3.2 SDDP (lower) selection function

3.3 U (upper) selection function

3.4 V (upper) selection function

Linear-polyhedral framework

A linear-polyhedral MSP is a MSP where the costs are polyhedral, *i.e.* their epigraph is a convex polyhedron, and the dynamics $f_t^w(x, u)$ are linear

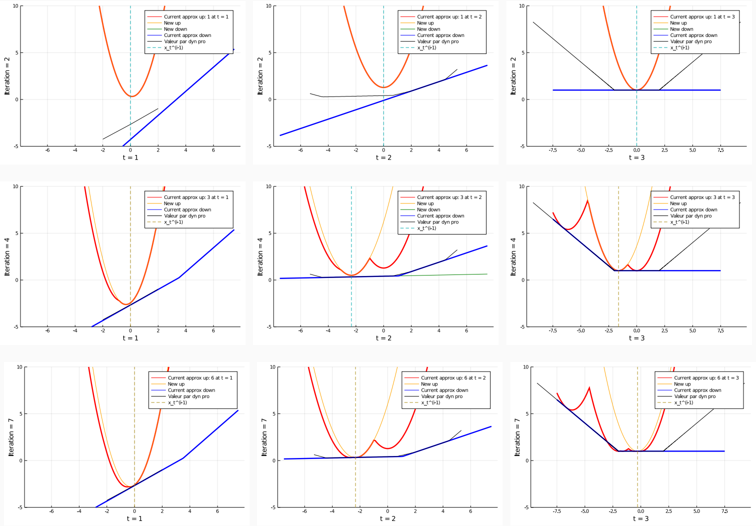
Proposition (Linear-polyhedral MSP are Lipschitz MSP)

Linear-polyhedral MSP are Lipschitz MSP

Proof.

The constraint mapping $\mathcal{U}_t^w$ has a convex polyhedral graph thus (*e.g.* [Rockafellar-Wets, Variational Analysis]) is Lipschitz with an explicit constant □

U-SDDP on a linear-polyhedral example



SDDP (lower) selection function

$\underline{S}_t^{\mathrm{SDDP}} : \mathcal{P}\left(\{\text{affine } L_{t+1}\text{-Lipschitz}\}\right) \times \mathbb{X} \to \{\text{affine } L_t\text{-Lipschitz}\}$

- Given $x_t \in \mathbb{X}$ and $\underline{F}_t \subset (\{\text{affine } L_{t+1}\text{-Lipschitz}\})$ finite, for each w, solve a LP:

$$\gamma = \mathcal{B}_t^w \left(\mathcal{V}_{\underline{F}_{t+1}} \right) (x_t) \quad \text{with} \quad \mathcal{V}_{\underline{F}_{t+1}} := \sup_{\underline{\phi}_{t+1} \in \underline{F}_{t+1}} \underline{\phi}_{t+1}$$

SDDP (lower) selection function

$\underline{S}_t^{\text{SDDP}} : \mathcal{P}(\{\text{affine } L_{t+1}\text{-Lipschitz}\}) \times \mathbb{X} \to \{\text{affine } L_t\text{-Lipschitz}\}$

- Given $x_t \in \mathbb{X}$ and $\underline{F}_{t+1} \subset (\{\text{affine } L_{t+1}\text{-Lipschitz}\})$ finite, for each w, solve a LP:

$$\gamma = \min_{x,\lambda,\mu} \lambda + \mu \quad \text{s.t.} \quad \forall i \in \underbrace{I_t}_{\text{finite}}, \langle c_t^{i,w}, (x; u)\rangle + \underbrace{d_t^{i,w}}_{\text{scalar}} \leq \lambda$$

$$\underbrace{\mathcal{V}_{\underline{F}_{t+1}}}_{\text{finite } \textbf{sup} \text{ of cuts}} \quad (A_t^w x + B_t^w u) \leq \mu$$

$$x = x_t \quad [\alpha]$$

- For each w, set $\underline{\phi}_w = \langle \alpha, \cdot - x_t\rangle + \gamma$ a tight at x_t and valid function $\underline{\phi}_w$ for $\mathcal{B}_t^w\left(\mathcal{V}_{\underline{F}_{t+1}}\right)$
- The average function $\underline{\phi} = \sum_w p_w \underline{\phi}_w$ is tight at x_t and valid for $\mathfrak{B}_t\left(\mathcal{V}_{\underline{F}_{t+1}}\right)$

U (upper) selection function

We build a Selection mapping for C-quadratics (or U-functions), *i.e.* of the form $\frac{C}{2}\|x - \mathrm{center}\|_2^2 + \mathrm{centerValue}$

$\overline{S}_t^{\mathrm{U}} : \mathcal{P}\left(\{C_{t+1}\text{-}quadratics\}\right) \times \mathbb{X} \to \{C_t\text{-quadratics}\}$

- Given $x_t \in \mathbb{X}$, $\overline{F}_t \subset (\{C_{t+1}\text{-quadratics}\})$, for each w, compute $\mathcal{B}_t^w\left(\mathcal{V}_{\overline{F}_{t+1}}\right)(x_t) = \min_{\overline{\phi}_j \in \overline{F}_{t+1}} \underbrace{\mathcal{B}_t^w\left(\overline{\phi}_j\right)(x_t)}_{\mathrm{QP}^2}$
- For each w, set $\overline{\phi}_w$ the C_t-quadratic such that $\overline{\phi}_w(x_t) = \min_j \gamma_j$ (attained at j^*) and $\overline{\phi}'_w(x_t) = \alpha_{j^*}$
- Here the min-additivity of $\mathcal{B}_t^w$ ensures the tightness of $\overline{\phi}_w$
- The average $\overline{\phi} = \sum_w p_w \overline{\phi}_w$, is a tight at x_t and valid function for $\mathfrak{B}_t\left(\mathcal{V}_{\overline{F}_{t+1}}\right)$

[2] $\gamma_j = \min_{x,\lambda} \lambda + \overline{\phi}_j\left(A_t^w x + B_t^w u\right)$ s.t. $\forall i \in I_t, \langle c_t^{i,w}, (x;u)\rangle + d_t^{i,w} \leq \lambda$ and $x = x_t$ $[\alpha_j]$

V (upper) selection function

We call *V-function* a function of the form $L\|x - \mathrm{center}\|_1 + \mathrm{centerValue}$ with given L.

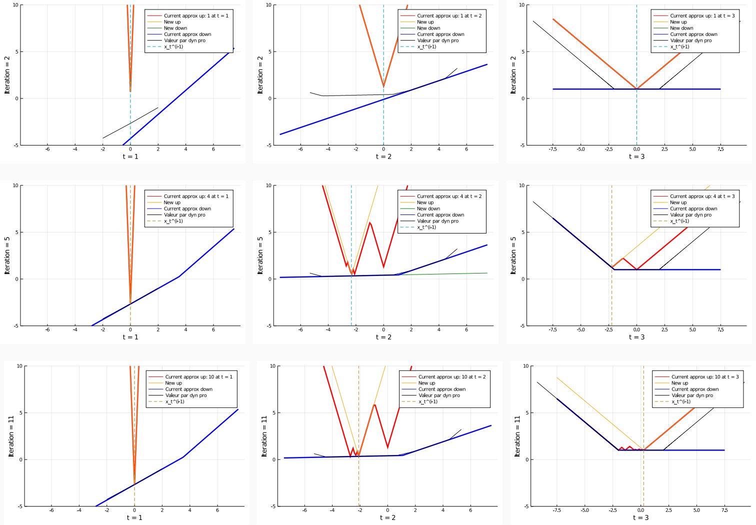
$\overline{S}_t^{\mathrm{V}} : \mathcal{P}\left(\{V\text{-functions with } L_{t+1}\}\right) \times \mathbb{X} \to \{V\text{-functions with } L_t\}$

- Given $x_t \in \mathbb{X}$, $\overline{F}_{t+1} \subset (\{V\text{-functions}\})$, denote by $\mathcal{V}_{\overline{F}_{t+1}}$ the finite infimum of the V-functions in $\overline{F}_t$. We solve a single LP

$$\gamma = \mathfrak{B}_t\left(\mathcal{V}_{\overline{F}_{t+1}}\right)(x_t)$$

- Set $\overline{\phi}$ the V-function such that $\mathrm{center} = x_t$ and $\mathrm{centerValue} = \gamma$
- The function $\overline{\phi}$ is tight at x_t and valid for $\mathfrak{B}_t\left(\mathcal{V}_{\overline{F}_{t+1}}\right)$ without the need of averaging

V-SDDP on a linear-polyhedral example



Conclusion

- TDP generates simultaneously monotonic approximations $\left(\underline{V}_t^k\right)_k$ and $\left(\overline{V}_t^k\right)_k$ of V_t
- Each approximation is either a min-plus or max-plus linear combinaisons of basic functions
- Each basic function should be tight and valid
- The approximations are refined iteratively along the Problem-child trajectory without discretizing the state space
- The gap between upper and lower approximation vanishes along the Problem-child trajectory
- Generalizes known approach of [Philpott and al. (2013)] and [Baucke and al. (2018)] for a variant of SDDP

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