Tropical Dynamic Programming for discrete time stochastic optimal control

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Outline

Multistage Stochastic optimization Problems (MSP)

Multistage Stochastic optimization Problem

$$\min_{(\mathsf{X},\mathsf{U})} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t^{\mathsf{W}_{t+1}}(\mathsf{X}_t, \mathsf{U}_t) + \psi(\mathsf{X}_T)\right]$$

$$\text{s.t. } \mathsf{X}_0 = x_0 \text{ given}, \forall t \in [\![0, T-1]\!]$$

$$\mathsf{X}_{t+1} = f_t^{\mathsf{W}_{t+1}}(\mathsf{X}_t, \mathsf{U}_t)$$

$$\sigma(\mathsf{U}_t) \subset \sigma(\mathsf{X}_0, \mathsf{W}_1, \dots, \mathsf{W}_{t+1}) \quad \text{(Hazard-Decision)}$$

Assumption (Finite support independent noises)

The sequence $(\mathsf{W}_t)_{t\in[\![1,T]\!]}$ is made of independent random variables each with finite support

MSP can be solved by Dynamic Programming

Bellman operators and Dynamic Programming

MSP can be solved by Dynamic Programming

- Pointwise Bellman operator

  for all $w \in \operatorname{supp}(\mathbf{W_{t+1}})$ and $\phi : \mathbb{X} \to \overline{\mathbb{R}}$

$$\mathcal{B}_t^w(\phi) : x \in \mathbb{X} \mapsto \min_u \Big(c_t^w(x, u) + \phi\big(f_t^w(x, u)\big) \Big) \in \overline{\mathbb{R}}$$

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- (Average) Bellman operator

$$\mathfrak{B}_t\,(\phi) : x \in \mathbb{X} \mapsto \mathbb{E}_{W_{t+1}}\left[\mathcal{B}_t^{\mathsf{W_{t+1}}}\,(\phi)\,(x)\right] \in \overline{\mathbb{R}}$$

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- (Average) Bellman operator

$$\mathfrak{B}_t(\phi) : x \in \mathbb{X} \mapsto \mathbb{E}_{W_{t+1}}\Big[\mathcal{B}_t^{\mathbf{W_{t+1}}}(\phi)(x)\Big] \in \overline{\mathbb{R}}$$

- Dynamic Programming Equations

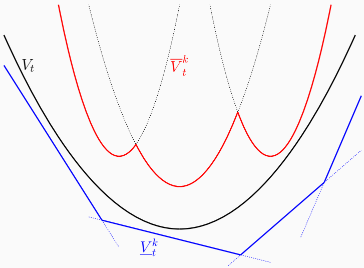
$$V_T = \psi \;\text{ and }\; \forall t \in [\![0, T-1]\!],\; V_t = \mathfrak{B}_t(V_{t+1})$$

Bellman operators and Dynamic Programming

MSP can be solved by Dynamic Programming

- Pointwise Bellman operator
  for all $w \in \mathrm{supp}\left(\mathbf{W_{t+1}}\right)$ and $\phi : \mathbb{X} \to \overline{\mathbb{R}}$
  $$\mathcal{B}_t^w\left(\phi\right) : x \in \mathbb{X} \mapsto \min_u \left(c_t^w\left(x, u\right) + \phi\left(f_t^w\left(x, u\right)\right)\right) \in \overline{\mathbb{R}}$$
- (Average) Bellman operator
  $$\mathfrak{B}_t\left(\phi\right) : x \in \mathbb{X} \mapsto \mathbb{E}_{W_{t+1}}\left[\mathcal{B}_t^{\mathbf{W_{t+1}}}\left(\phi\right)\left(x\right)\right] \in \overline{\mathbb{R}}$$
- Dynamic Programming Equations
  $$V_T = \psi \ \text{ and } \ \forall t \in [\![0, T-1]\!], \ V_t = \mathfrak{B}_t\left(V_{t+1}\right)$$
    - V_t is called the value function at time $t \in [\![0, T]\!]$
- The value of MSP is equal to $V_0\left(x_0\right)$

Build an algorithm that simultaneously generates upper and lower approximations of V_t as min-plus linear and max-plus linear combinations of basic functions



For all $t \in [\![0, T]\!]$, construct increasing sequences of basic functions $\left(\underline{F}_t^k\right)_{k\in\mathbb{N}}$ and $\left(\overline{F}_t^k\right)_{k\in\mathbb{N}}$

$$\begin{cases} \underline{V}_t^k = \sup_{\underline{\phi}\in\underline{F}_t^k} \underline{\phi} \\ \overline{V}_t^k = \inf_{\overline{\phi}\in\overline{F}_t^k} \overline{\phi} \end{cases}$$

Min-plus & Max-plus approximations of V_t

Build an algorithm that simultaneously generates upper and lower approximations of V_t as min-plus linear and max-plus linear combinations of basic functions

- Generalizes the Min-plus algorithm for deterministic control problems (McEneaney 2007, Qu 2014) giving upper approximations as infima of quadratics
- and the Stochastic Dual Dynamic Programming (SDDP) algorithm (Pereira and Pinto 1991, Shapiro 2011, ...) giving lower approximations as suprema of affine cuts

Lipschitz Multistage Stochastic optimization Problems

Assumption (Lipschitz dynamic, costs and constraints)

For every time $t < T$ and $w \in \mathrm{supp}\left(\mathbf{W_{t+1}}\right)$,

- *dynamics f_t^w are Lipschitz continuous*
- *cost c_t^w are Lipschitz continuous on $\mathrm{dom}\ c_t^w$*
- *constraint set-valued mapping $\mathcal{U}_t^w$ is Lipschitz continuous on X_t,*

$$d_{\mathcal{H}}\left(\mathcal{U}_t^w\left(x_1\right), \mathcal{U}_t^w\left(x_2\right)\right) \leq L_{\mathcal{U}_t^w}\|x_1 - x_2\|$$

Proposition (Lipschitz MSP implies regularity of $\mathfrak{B}_t$)

If $V : \mathbb{X} \to \overline{\mathbb{R}}$ L_{t+1}-Lipschitz on X_{t+1},
then $\mathfrak{B}_t\left(V\right)$ is L_t-Lipschitz on X_t for some constant $L_t > 0$ which only depends on the data of the MSP problem and L_{t+1}.

Constraint set-valued mapping

For each noise $w \in \operatorname{supp}(\mathbf{W_{t+1}})$, $t \in [\![0, T-1]\!]$, define the constraint set-valued mapping $\mathcal{U}_t^w : \mathbb{X} \rightrightarrows \mathbb{U}$

$$\mathcal{U}_t^w(x) := \{u \in \mathbb{U} \mid c_t^w(x, u) < +\infty \text{ and } f_t^w(x, u) \in X_{t+1}\}.$$ [1]

Assumption (Recourse assumption)

The set-valued mapping $\mathcal{U}_t^w$ is non-empty compact valued

Proposition (Known domains of V_t)

Under the recourse assumption, $\operatorname{dom} V_t = X_t$

[1] $\forall w \in \operatorname{supp}(\mathbf{W_{t+1}})$, $X_t^w := \pi_{\mathbb{X}}\left(\operatorname{dom} c_t^w\right)$, and $X_t := \cap_{w \in \operatorname{supp}(\mathbf{W_{t+1}})} X_t^w$.

Section content

How to select new basic functions ?

Input: sequence $(x_t)_{t \in [\![0,T]\!]}$ of trial points, sequence $(F_t)_{t \in [\![0,T]\!]}$ of sets of basic functions

Output: sequence $(\phi_t)_{t \in [\![0,T]\!]}$ of basic functions

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Output: sequence $(\phi_t)_{t\in[\![0,T]\!]}$ of basic functions

Case $t = T$

Tightness Assumption (local property)

$$\phi_T\left(x_T\right) = V_T\left(x_T\right)$$

Validity Assumption (global property)

$$\phi_T \geq V_T \quad \text{(Min-plus lin. combinations case)}$$

$$\phi_T \leq V_T \quad \text{(Max-plus lin. combinations case)}$$

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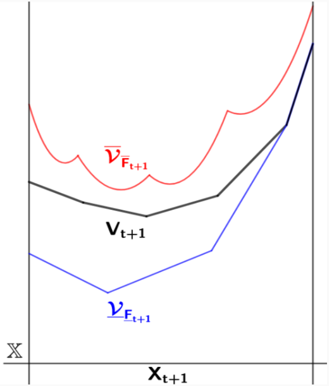
Notation: $\mathcal{V}_{F_{t+1}}$ the sup or inf of basic functions in F_{t+1}



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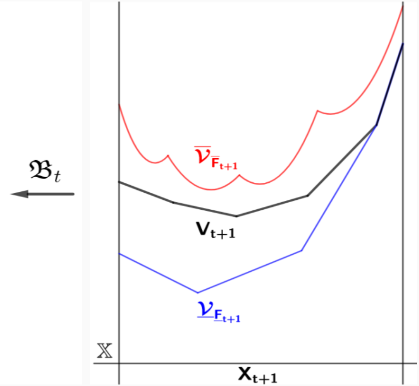

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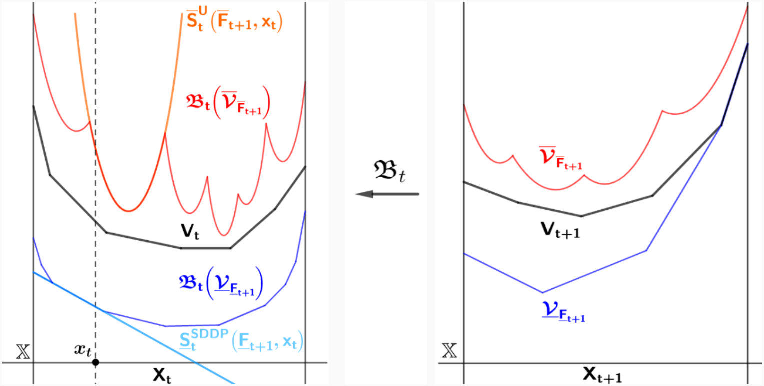

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Output: sequence $(\phi_t)_{t\in[\![0,T]\!]}$ of basic functions

<u>Case $t < T$</u>

Tightness Assumption (local property)

$$\phi_t\left(x_t\right) = \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right)\left(x_t\right)$$

Validity Assumption (global property)

$$\phi_t \geq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right) \quad \text{(Min-plus lin. combinations case)}$$

$$\phi_t \leq \mathfrak{B}_t\left(\mathcal{V}_{F_{t+1}}\right) \quad \text{(Max-plus lin. combinations case)}$$

Problem-child trajectory of Baucke and al. (2018)

Input: two sequences of functions $\underline{V}_0, \dots, \underline{V}_T$ and $\overline{V}_0, \dots, \overline{V}_T$

Output: Problem-child trajectory, states $(x_0^*, \dots, x_T^*)$.

Initial state x_0^* is given, then **for** $t < T$

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Initial state x_0^* is given, then **for** $t < T$

1. For all $w \in \mathrm{supp}\,(\mathbf{W_{t+1}})$, compute optimal control at x_t^*

$$u_t^w \in \operatorname*{arg\,min}_{u \in U} \Big(c_t^w\,(x_t^*, u) + \underline{V}_{t+1}\big(f_t^w\,(x_t^*, u)\,\big)\Big)$$

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2. Compute "the worst" noise
$w^* \in \arg\max_{w \in \mathbf{W_{t+1}}} \big(\overline{V}_{t+1} - \underline{V}_{t+1}\big)\big(f_t^w\,(x_t^*, u_t^w)\,\big)$

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3. Set $x_{t+1}^* = f_t^{w^*}\big(x_t^*, u_t^{w^*}\big)$

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3. Set $x_{t+1}^* = f_t^{w^*}\big(x_t^*, u_t^{w^*}\big)$

Interpretation

Problem child trajectory = "Worst" optimal trajectory of the lower approximations

Algorithm 1 Tropical Dynamic Programming (TDP)

Input: Selection functions and $(\mathbf{W_t})_{t\in[\![1,T]\!]}$ independent r.v. with finite support.

Output: Sequence of sets $\left(\overline{F}_t^k\right)_{k\in\mathbb{N}}$, $\left(\underline{F}_t^k\right)_{k\in\mathbb{N}}$

Tropical Dynamic Programming (TDP) algorithm

Algorithm 2 Tropical Dynamic Programming (TDP)

Input: Selection functions and $(\mathsf{W_t})_{t \in [\![1,T]\!]}$ independent r.v. with finite support.

Output: Sequence of sets $\left(\overline{F}_t^k\right)_{k \in \mathbb{N}}$, $\left(\underline{F}_t^k\right)_{k \in \mathbb{N}}$

1: For every $t \in [\![0,T]\!]$, $\overline{F}_t^0 := \emptyset$ and $\underline{F}_t^0 := \emptyset$

Tropical Dynamic Programming (TDP) algorithm

Algorithm 3 Tropical Dynamic Programming (TDP)

Input: Selection functions and $(\mathbf{W_t})_{t \in [\![1,T]\!]}$ independent r.v. with finite support.

Output: Sequence of sets $\left(\overline{F}_t^k\right)_{k \in \mathbb{N}}$, $\left(\underline{F}_t^k\right)_{k \in \mathbb{N}}$

For every $t \in [\![0,T]\!]$, $\overline{F}_t^0 := \emptyset$ and $\underline{F}_t^0 := \emptyset$

for $k \geq 0$ **do**

  Forward. Compute Problem-child trajectory $\left(x_t^k\right)_{t \in [\![0,T]\!]}$ using $\overline{V}_t^k = \inf_{\overline{\phi} \in \overline{F}_t^k} \overline{\phi}$ and $\underline{V}_t^k = \sup_{\underline{\phi} \in \underline{F}_t^k} \underline{\phi}$

end for

Tropical Dynamic Programming (TDP) algorithm

Algorithm 4 Tropical Dynamic Programming (TDP)

Input: Selection functions and $(\mathrm{W_t})_{t\in[\![1,T]\!]}$ independent r.v. with finite support.

Output: Sequence of sets $\left(\overline{F}_t^k\right)_{k\in\mathbb{N}}$, $\left(\underline{F}_t^k\right)_{k\in\mathbb{N}}$

1: For every $t \in [\![0,T]\!]$, $\overline{F}_t^0 := \emptyset$ and $\underline{F}_t^0 := \emptyset$

2: **for** $k \geq 0$ **do**

3: Forward. Compute Problem-child trajectory $\left(x_t^k\right)_{t\in[\![0,T]\!]}$ using $\overline{V}_t^k = \inf_{\overline{\phi}\in\overline{F}_t^k} \overline{\phi}$ and $\underline{V}_t^k = \sup_{\underline{\phi}\in\underline{F}_t^k} \underline{\phi}$

4: Backward. Compute new basic functions $\left(\overline{\phi}_t\right)_{t\in[\![0,T]\!]}$ and $\left(\underline{\phi}_t\right)_{t\in[\![0,T]\!]}$ and update $\overline{F}_t^{k+1} := \overline{F}_t^k \cup \left\{\overline{\phi}_t\right\}$ and $\underline{F}_t^{k+1} := \underline{F}_t^k \cup \left\{\underline{\phi}_t\right\}$, $t \in [\![0,T]\!]$

5: **end for**

Section content

Convergence to limits $\underline{V}_t^*$ and $\overline{V}_t^*$

Under finite independent noises, Lipschitz data and recourse assumptions we have

Existence of an approximating limit

The sequence of functions $\left(\underline{V}_t^k\right)_{k\in\mathbb{N}}$ (resp. $\left(\overline{V}_t^k\right)_{k\in\mathbb{N}}$) generated by TDP converges uniformly on every compact set included in the domain of V_t to a function $\underline{V}_t^*$ (resp. $\overline{V}_t^*$).

Some features of TDP

Convergence to limits $\underline{V}_t^*$ and $\overline{V}_t^*$

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Some features of TDP

- No need to discretize the state space
- $\left(\underline{V}_t^k\right)_k$ and $\left(\overline{V}_t^k\right)_k$ are monotonic

Convergence to limits $\underline{V}_t^*$ and $\overline{V}_t^*$

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The sequence of functions $\left(\underline{V}_t^k\right)_{k\in\mathbb{N}}$ (resp. $\left(\overline{V}_t^k\right)_{k\in\mathbb{N}}$) generated by TDP converges uniformly on every compact set included in the domain of V_t to a function $\underline{V}_t^*$ (resp. $\overline{V}_t^*$).

Some features of TDP

- No need to discretize the state space
- $\left(\underline{V}_t^k\right)_k$ and $\left(\overline{V}_t^k\right)_k$ are monotonic
- $\underline{V}_t^*$ and $\overline{V}_t^*$ are close to V_t on “interesting points”, but may be far from V_t elsewhere.

Asymptotic convergence of TDP

Under finite independent noises, Lipschitz data and recourse assumptions we have

Convergence of TDP [Akian, Chancelier, T., 2020]

Denote by $\left(x_t^k\right)_{0 \leq t \leq T}$ the k-th Problem-child trajectory.

For every accumulation point x_t^* of $\left(x_t^k\right)_{k \in \mathbb{N}}$, we have

$$\overline{V}_t^k\left(x_t^k\right) - \underline{V}_t^k\left(x_t^k\right) \underset{k \to +\infty}{\longrightarrow} 0 \quad \text{and} \quad \overline{V}_t^*\left(x_t^*\right) = V_t\left(x_t^*\right) = \underline{V}_t^*\left(x_t^*\right)$$

This result generalizes the convergence of SDDP *à la* [Philpott and al. (2013)] and [Baucke and al. (2018)] seen as a specific instance of TDP for the linear-polyhedral framework

Idea of the proof, details in [Akian, Chancelier, T., 2020]

- $\left(\underline{V}_t^k\right)_k$ (resp. $\left(\overline{V}_t^k\right)_k$) converges uniformly to $\underline{V}_t^*$ (resp. $\overline{V}_t^*$) on the domain of V_t by Arzela-Ascoli theorem

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- Exploiting monotonicity of the approximations and that each operator $\mathcal{B}_t^w$ is order preserving

$$
\begin{aligned}
0 &\leq \overline{V}_t^{k+1}\left(x_t^k\right) - \underline{V}_t^{k+1}\left(x_t^k\right) \\
&\leq \sum_{w \in \mathrm{supp}(\mathbf{W}_{t+1})} \mathbb{P}\left[\mathbf{W}_{t+1} = w\right] \left[\left(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\right)\left(f_t^w\left(x_t^k, u_t^k(w)\right)\right)\right]
\end{aligned}
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\end{aligned}
$$

- PC-trajectory is the “worst” optimal trajectory

$$
0 \leq \overline{V}_t^{k+1}\left(x_t^k\right) - \underline{V}_t^{k+1}\left(x_t^k\right) \leq \overline{V}_{t+1}^k\left(x_{t+1}^k\right) - \underline{V}_{t+1}^k\left(x_{t+1}^k\right)
$$

Idea of the proof, details in [Akian, Chancelier, T., 2020]

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$$

- Taking the limit in k

$$
0 \leq \overline{V}_t^*\left(x_t^*\right) - \underline{V}_t^*\left(x_t^*\right) \leq \overline{V}_{t+1}^*\left(x_{t+1}^*\right) - \underline{V}_{t+1}^*\left(x_{t+1}^*\right)
$$

Idea of the proof, details in [Akian, Chancelier, T., 2020]

- $\left(\underline{V}_t^k\right)_k$ (resp. $\left(\overline{V}_t^k\right)_k$) converges uniformly to $\underline{V}_t^*$ (resp. $\overline{V}_t^*$) on the domain of V_t by Arzela-Ascoli theorem
- Exploiting monotonicity of the approximations and that each operator $\mathcal{B}_t^w$ is order preserving

$$0 \leq \overline{V}_t^{k+1}\left(x_t^k\right) - \underline{V}_t^{k+1}\left(x_t^k\right)$$
$$\leq \sum_{w \in \mathrm{supp}(\mathbf{W}_{t+1})} \mathbb{P}\left[\mathbf{W}_{t+1} = w\right]\left[\left(\overline{V}_{t+1}^k - \underline{V}_{t+1}^k\right)\left(f_t^w\left(x_t^k, u_t^k(w)\right)\right)\right]$$

- PC-trajectory is the “worst” optimal trajectory

$$0 \leq \overline{V}_t^{k+1}\left(x_t^k\right) - \underline{V}_t^{k+1}\left(x_t^k\right) \leq \overline{V}_{t+1}^k\left(x_{t+1}^k\right) - \underline{V}_{t+1}^k\left(x_{t+1}^k\right)$$

- Taking the limit in k

$$0 \leq \overline{V}_t^*\left(x_t^*\right) - \underline{V}_t^*\left(x_t^*\right) \leq \overline{V}_{t+1}^*\left(x_{t+1}^*\right) - \underline{V}_{t+1}^*\left(x_{t+1}^*\right)$$

- Conclude by backward recursion on t

Linear-polyhedral framework

Linear dynamics $(x, u) \mapsto f_t^w(x, u)$

Polyhedral costs $(x, u) \mapsto c_t^w(x, u)$ (convex polyhedral epigraph)

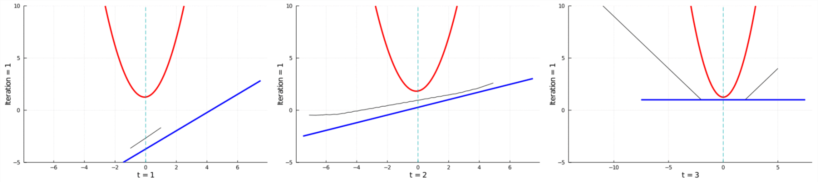
Proposition (Linear-polyhedral MSP are Lipschitz MSP)

Linear-polyhedral MSP are Lipschitz MSP

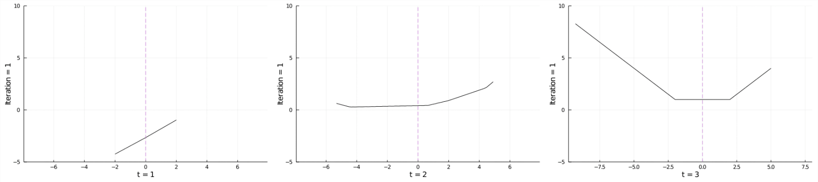
Proof.

The constraint mapping $\mathcal{U}_t^w$ has a convex polyhedral graph thus (*e.g.* [Rockafellar-Wets, Variational Analysis]) is Lipschitz with an explicit constant □

U-SDDP on a linear-polyhedral example



V-SDDP on a linear-polyhedral example



Complexity of TDP

- G. Lan[2] obtained complexity of SDDP (and EDDP) in 2020
  Precision of $T\epsilon$ archived after at most $T(\frac{D}{\epsilon}+1)^N$ iterations
  D diameter state spaces
  N dimension state/control (decision) space

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- For TDP, overall complexity depends on the complexity of computing basic functions

Selection mapping	Computational difficulty
SDDP	$\mathrm{Card}\left(\mathsf{W}_{t+1}\right)$ LPs
U	$\mathrm{Card}\left(\mathsf{W}_{t+1}\right) \cdot \mathrm{Card}\left(F\right)$ QPs
V	one LP

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Conclusion

- Monotonic approximations $\left(\overline{V}_t^k\right)_k$ and $\left(\underline{V}_t^k\right)_k$ of V_t

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- Additional results (deterministic case) in [Akian, Chancelier, T. (2018)]

Perspectives and ongoing works

- Allow approximations to only be monotonous at the trial points (to do pruning)
- (Level) regularization of TDP
- Compute sharp(er) bounds of Lipschitz regularity of $\mathfrak{B}_t$
- Extensive numerical comparisons in higher dimensions
- More flexibility on the choice of the trial points : randomly drawn and satisfy a sufficient condition to get convergence. Already done in the deterministic case.
- Handle more complex noise structures

References

References

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Thank you !